

Simple and large equivalence relations

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Abstract

We construct ergodic discrete probability measure preserving equivalence relations \mathcal{R} that has no proper ergodic normal subequivalence relations and no proper ergodic finite-index subequivalence relations. We show that every treeable equivalence relation satisfying a mild ergodicity condition and cost > 1 surjects onto every countable group with ergodic kernel. Lastly, we provide a simple characterization of normality for subequivalence relations and an algebraic description of the quotient.

Keywords: ergodic equivalence relations

MSC: 37A20, 37A15

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1 Introduction

Let (X, μ) denote a standard Borel probability space and $\mathcal{R} \subset X \times X$ a Borel equivalence relation. We say that \mathcal{R} is **discrete** if for all $x \in X$, the \mathcal{R} -class of x , denoted $[x]_{\mathcal{R}}$, is countable. All equivalence relations considered in this note are discrete regardless of whether this is mentioned explicitly. We endow \mathcal{R} with two measures μ_L and μ_R satisfying:

$$\mu_L(S) = \int |S_x| d\mu(x), \quad \mu_R(S) = \int |S^y| d\mu(y)$$

where

$$S_x = \{y \in X : (x, y) \in S\}, \quad S^y = \{x \in X : (x, y) \in S\}.$$

We say that μ is **\mathcal{R} -quasi-invariant** if μ_L and μ_R are in the same measure class. We say μ is **\mathcal{R} -invariant** or \mathcal{R} is **measure-preserving** if $\mu_L = \mu_R$. A subset $A \subset X$ is **\mathcal{R} -invariant** or **\mathcal{R} -saturated** if it is a union of \mathcal{R} -classes. We say \mathcal{R} is **ergodic** if for every measurable \mathcal{R} -invariant subset $A \subset X$, $\mu(A) \in \{0, 1\}$. In the sequel we will always assume μ is \mathcal{R} -quasi-invariant.

A Borel subset $\mathcal{S} \subset \mathcal{R}$ is a **subequivalence relation** if it is a Borel equivalence relation in its own right. If $\mathcal{S}, \mathcal{S}' \subset \mathcal{R}$ are two subequivalence relations whose symmetric difference is null with respect to μ_L (or equivalently μ_R) then we say \mathcal{S} and \mathcal{S}' agree almost everywhere (a.e.). From here on we will not distinguish between relations that agree almost everywhere. We say \mathcal{S} is **proper** if it does not equal \mathcal{R} a.e. We usually write $\mathcal{S} \leq \mathcal{R}$ to mean $\mathcal{S} \subset \mathcal{R}$ (a.e.) when \mathcal{S} is a subequivalence relation.

The concept of a normal subequivalence relation was introduced in [FSZ88, FSZ89] where it was shown that if $\mathcal{S} \subset \mathcal{R}$ is normal then there is a natural quotient object, denoted \mathcal{R}/\mathcal{S} , which is a discrete measured groupoid. Moreover, if \mathcal{S} is ergodic then \mathcal{R}/\mathcal{S} is a countable group and in this case we say \mathcal{R} **surjects** onto \mathcal{R}/\mathcal{S} and \mathcal{R}/\mathcal{S} is a quotient of \mathcal{R} .

Unfortunately, the definition of normality given in [FSZ88, FSZ89] is rather complicated. In §2 we provide a simple characterization: \mathcal{S} is **normal** if and only if it is the kernel of a Borel morphism $c : \mathcal{R} \rightarrow \mathcal{G}$ where \mathcal{G} is a discrete Borel groupoid. We also show in §?? that when \mathcal{S} is normal and ergodic in \mathcal{R} then \mathcal{R}/\mathcal{S} is isomorphic with the full group $[\mathcal{R}]$ modulo the normalizer of $[\mathcal{S}]$ in $[\mathcal{R}]$. This fact was probably known to the authors of [FSZ88, FSZ89] but it is not explicitly stated.

If $\mathcal{S} \subset \mathcal{R}$ is an arbitrary subequivalence relation and \mathcal{R} is ergodic then, as shown in [FSZ89], there exists a number $N \in \mathbb{N} \cup \{\infty\}$ such that for a.e. $x \in X$, $[x]_{\mathcal{R}}$ contains exactly N \mathcal{S} -classes. This number N is called the **index** of \mathcal{S} in \mathcal{R} and is denoted $N = [\mathcal{R} : \mathcal{S}]$. Our first main result:

Theorem 1.1. *There exists an ergodic discrete probability-measure-preserving equivalence relation \mathcal{R} such that \mathcal{R} does not contain any proper ergodic normal subequivalence relations. Moreover, \mathcal{R} does not contain any proper finite-index ergodic subequivalence relations.*

The proof of Theorem 1.1 is based on Popa's Cocycle Superrigidity Theorem [Po07]. The ergodicity condition is necessary because if \mathcal{P} is any finite measurable partition of X then

the subequivalence relation \mathcal{S} defined by: $(x, y) \in \mathcal{S}$ if and only $(x, y) \in \mathcal{R}$ and x, y are in the same part of \mathcal{P} , has finite-index and is normal in \mathcal{R} . Of course, \mathcal{S} is not ergodic if \mathcal{P} is nontrivial.

Remark 1. Stefaan Vaes constructed the first explicit examples of type II_1 von Neumann algebras having only trivial finite index subfactors in [Va08] by a twisted group-measure space construction. Moreover it follows from [Va08, Theorem 6.4] that the orbit-equivalence relation of the generalized Bernoulli shift action $SL(2, \mathbb{Q}) \ltimes \mathbb{Q}^2 \curvearrowright (X_0, \mu_0)^{\mathbb{Q}^2}$ has no finite index ergodic subequivalence relations and no nontrivial finite extensions. Here (X_0, μ_0) is any nontrivial atomic probability space with unequal weights (so that it has trivial automorphism group). More generally, the proof of [Va08, Theorem 6.4] shows how to describe all finite index subequivalence relations, extensions and bimodules whenever cocycle super-rigidity applies.

We say that a measured equivalence relation \mathcal{R} is **large** if for every countable group G there exists an ergodic normal subequivalence relation $\mathcal{N} \leq \mathcal{R}$ such that $\mathcal{R}/\mathcal{N} \cong G$.

Next we prove that some treeable equivalence relations are large:

Theorem 1.2. *Suppose \mathcal{R} is a treeable ergodic equivalence relation on (X, μ) of cost > 1 and there exists an ergodic primitive proper subequivalence relation $\mathcal{S} \leq \mathcal{R}$. Then \mathcal{R} is large.*

The terms treeable and primitive are explained in §6 below. (Primitive means the same as free factor; this notion was studied by Damien Gaboriau [Ga00, Ga05]). For example, the orbit-equivalence relation of any Bernoulli shift over a rank ≥ 2 free group satisfies the hypothesis above and hence is large. It is an open question whether every ergodic treeable equivalence relation with cost > 1 satisfies the hypotheses of Theorem 1.2. In particular it is unknown whether every such equivalence relation surjects every countable group. In unpublished work, Clinton Conley, Damien Gaboriau, Andrew Marks and Robin Tucker-Drob have proven that any treeable *strongly ergodic* pmp equivalence relation satisfies the hypothesis of Theorem 1.2 and therefore surjects onto every countable group.

Organization: In §2 we provide a simple characterization for normality of a subequivalence relation. §3 provides an algebraic description for the quotient \mathcal{R}/\mathcal{N} . §4 reviews generalized Bernoulli shifts and Popa's Cocycle Superrigidity Theorem [Po07]. The latter is used in §5 to prove Theorem 1.1. The last section §6 proves Theorem 1.2. The proof is mostly independent of the rest of the paper.

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2 A simple criterion for normality

Definition 1 (Choice functions). Let \mathcal{R} be an ergodic discrete Borel equivalence relation on (X, μ) . Let $\mathcal{S} \subset \mathcal{R}$ be a Borel subequivalence relation and let $N = [\mathcal{R} : \mathcal{S}]$. A **family of choice functions** is a set $\{\phi_j\}_{j=1}^N$ of Borel functions $\phi_j : X \rightarrow X$ such that for each $x \in X$, $\{[\phi_j(x)]_{\mathcal{S}}\}_{j=1}^N$ is a partition of $[x]_{\mathcal{R}}$. [FSZ89, Lemma 1.1] shows that a family of choice functions exists.

Theorem 2.1. *Let \mathcal{R} be an ergodic discrete Borel equivalence relation on (X, μ) . Let $\mathcal{S} \subset \mathcal{R}$ be a subequivalence relation. The following are equivalent.*

1. \mathcal{S} is normal in \mathcal{R} in the sense of [FSZ89, Definition 2.1].
2. There are choice functions $\{\phi_j\}$ for $\mathcal{S} \subset \mathcal{R}$ with $\phi_j \in \text{End}_{\mathcal{R}}(\mathcal{S})$ for all j . This means that if $(x, y) \in \mathcal{S}$ then $(\phi_j(x), \phi_j(y)) \in \mathcal{S}$.
3. The extension $\rho : \hat{\mathcal{R}} = \mathcal{S} \times_{\sigma} J \rightarrow \mathcal{S}$ is normal in the sense of Zimmer [Zi76];
4. There is a discrete, ergodic measured groupoid (\mathcal{H}, ν) and a homomorphism $\theta : \mathcal{R} \rightarrow \mathcal{H}$ such that
 - (a) $\ker(\theta) = \mathcal{S}$;
 - (b) θ is class-surjective in the following sense: for any $h \in \mathcal{H}$ and $x \in X$ with $\theta(x)$ equal to the source of h , there exists $y \in [x]_{\mathcal{R}}$ with $\theta(y, x) = h$;
 - (c) for any discrete ergodic measured groupoid (\mathcal{H}', ν') and homomorphism $\theta' : \mathcal{R} \rightarrow \mathcal{H}'$ with $\ker(\theta') \supset \mathcal{S}$ there is a homomorphism $\kappa : \mathcal{H} \rightarrow \mathcal{H}'$ with $\kappa\theta = \theta'$;
5. there is a discrete Borel groupoid \mathcal{G} and a Borel homomorphism $c : \mathcal{R} \rightarrow \mathcal{G}$ with $\mathcal{S} = \text{Ker}(c)$.

Proof. The equivalence of the first four statements is [FSZ89, Theorem 2.2]. Clearly (4) implies (5). So we need only show that (5) implies (2). So let \mathcal{G} be a discrete Borel groupoid with unit space \mathcal{G}^0 , source and range maps $\mathfrak{s}, \mathfrak{r} : \mathcal{G} \rightarrow \mathcal{G}^0$. As in the proof of the Feldman-Moore Theorem [FM77, Theorem 1], there exists a countable family of Borel functions $\{f_j\}_{j=1}^{\infty}$, $f_j : \mathcal{G}^0 \rightarrow \mathcal{G}$ such that for every $x \in \mathcal{G}^0$,

$$\{f_j(x)\}_{j \in \mathbb{N}} = \mathfrak{s}^{-1}(x).$$

Let $c : \mathcal{R} \rightarrow \mathcal{G}$ be a Borel homomorphism. Because \mathcal{R} is ergodic there exists $n \in \mathbb{N} \cup \{\infty\}$ be such that $\{c(x, y) : y \in [x]_{\mathcal{R}}\}$ has cardinality n for a.e. x . Let $\psi_j : X \rightarrow X$ be a Borel map such that if $\psi_j(x) = y$ then $y \in [x]_{\mathcal{R}}$ and $c(x, y) = f_j(c(x, y))$.

For each $x \in X$, define $\phi_1(x) = \psi_1(x)$. For $i > 1$ inductively define $\phi_i(x) = \psi_j(x)$ where j is the smallest number such that there does not exist $k < i$ with $\phi_k(x) = \psi_j(x)$. Then $\{\phi_j\}_{j=1}^n$ is family of choice functions satisfying (2). □

3 The quotient group

The purpose of this section is to provide an algebraic description of the quotient \mathcal{R}/\mathcal{N} when $\mathcal{N} \triangleleft \mathcal{R}$ is normal.

To be precise, let \mathcal{R} denote an ergodic probability-measure-preserving discrete equivalence relation on a probability space (X, μ) . Let $\text{Aut}(X, \mu)$ be the group of all measure-preserving automorphisms $\phi : X \rightarrow X$. We implicitly identify two automorphisms that agree almost everywhere. Let $\text{Aut}(\mathcal{R})$ be the subgroup of all $\phi \in \text{Aut}(X, \mu)$ such that $x\mathcal{R}y \Rightarrow \phi(x)\mathcal{R}\phi(y)$ (for μ_L -a.e. (x, y)). Also let $[\mathcal{R}] = \text{Inn}(\mathcal{R})$ be the subgroup of all $\phi \in \text{Aut}(\mathcal{R})$ such that $x\mathcal{R}\phi(x)$ for a.e. x . Then $[\mathcal{R}]$ is normal in $\text{Aut}(\mathcal{R})$, so we may consider the quotient $\text{Out}(\mathcal{R}) := \text{Aut}(\mathcal{R})/[\mathcal{R}]$.

In the sequel we use the word ‘countable’ to mean ‘countable or finite’.

Let Γ be a countable subgroup of $\text{Aut}(\mathcal{R})$. Let $\mathcal{R}_\Gamma := \langle \mathcal{R}, \Gamma \rangle$ denote the smallest equivalence relation on X such that $\mathcal{R} \subset \mathcal{R}_\Gamma$ and $(x, \gamma x) \in \mathcal{R}_\Gamma$ for all $x \in X$ and $\gamma \in \Gamma$. Observe that this is a discrete probability-measure-preserving ergodic equivalence relation and \mathcal{R} is normal in \mathcal{R}_Γ .

Lemma 3.1. *If $\Gamma, \Lambda \leq \text{Aut}(\mathcal{R})$ are countable subgroups and $\Gamma[\mathcal{R}] = \Lambda[\mathcal{R}]$ then $\mathcal{R}_\Gamma = \mathcal{R}_\Lambda$.*

Proof. This is straightforward. □

So if $\Gamma \leq \text{Out}(\mathcal{R})$ is any countable subgroup then we may define $\mathcal{R}_\Gamma := \mathcal{R}_{\Gamma'}$ where $\Gamma' \leq \text{Aut}(\mathcal{R})$ is any countable subgroup such that $\Gamma = \Gamma'[\mathcal{R}]$. The next lemma follows immediately.

Lemma 3.2. *If $\Gamma \leq \Lambda \leq \text{Out}(\mathcal{R})$ are countable subgroups then $\mathcal{R}_\Gamma \leq \mathcal{R}_\Lambda$.*

Theorem 3.3. *Let $\mathcal{R} \leq \mathcal{U}$ be ergodic discrete probability-measure-preserving equivalence relations on (X, μ) . Suppose \mathcal{R} is normal in \mathcal{U} . Then there exists a countable subgroup $\Gamma \leq \text{Out}(\mathcal{R})$ such that $\mathcal{U} = \mathcal{R}_\Gamma$. Moreover, $\mathcal{R}_\Gamma/\mathcal{R}$ is isomorphic to Γ . In particular, $[\mathcal{R}_\Gamma : \mathcal{R}] = |\Gamma|$.*

Proof. This follows from [FSZ89, Theorems 2.12 and 2.13]. □

Corollary 3.4. *Let $\mathcal{R} \leq \mathcal{U}$ be ergodic discrete probability-measure-preserving equivalence relations on (X, μ) . Suppose \mathcal{R} is normal in \mathcal{U} . Then*

$$\mathcal{U}/\mathcal{R} \cong N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}]$$

where

$$N_{\mathcal{U}}(\mathcal{R}) := \{g \in [\mathcal{U}] : g[\mathcal{R}]g^{-1} = [\mathcal{R}]\}.$$

Moreover $N_{\mathcal{U}}(\mathcal{R}) = [\mathcal{U}] \cap \text{Aut}(\mathcal{R})$.

Proof. We prove the last claim first. So suppose $g \in N_{\mathcal{U}}(\mathcal{R})$ and $x \in X$. Then for any $\phi \in [\mathcal{R}]$ we must have $g\phi g^{-1} \in [\mathcal{R}]$. This implies $(x, g\phi g^{-1}x) \in \mathcal{R}$ which implies, by replacing x with gx , that $(gx, g\phi x) \in \mathcal{R}$. Since ϕ is arbitrary and $[\mathcal{R}]$ acts transitively on each \mathcal{R} -class, this implies $g \in \text{Aut}(\mathcal{R})$. Thus $N_{\mathcal{U}}(\mathcal{R}) \subset [\mathcal{U}] \cap \text{Aut}(\mathcal{R})$.

Now suppose $g \in [\mathcal{U}] \cap \text{Aut}(\mathcal{R})$. If $\phi \in [\mathcal{R}]$ then $(x, \phi x) \in \mathcal{R}$. This implies $(gx, g\phi x) \in \mathcal{R}$. By replacing x with $g^{-1}x$ we obtain $(x, g\phi g^{-1}x) \in \mathcal{R}$ which implies $g\phi g^{-1} \in [\mathcal{R}]$ (since x is arbitrary). Thus $g \in N_{\mathcal{U}}(\mathcal{R})$. This proves $N_{\mathcal{U}}(\mathcal{R}) = [\mathcal{U}] \cap \text{Aut}(\mathcal{R})$.

By Theorem 3.3 there exists a countable subgroup $\Gamma \leq \text{Out}(\mathcal{R})$ such that $\mathcal{R}_{\Gamma} = \mathcal{U}$ and $\mathcal{U}/\mathcal{R} \cong \Gamma$. Let $\tilde{\Gamma} \leq \text{Aut}(\mathcal{R})$ be the inverse image of Γ under the quotient map $\text{Aut}(\mathcal{R}) \rightarrow \text{Aut}(\mathcal{R})/[\mathcal{R}] = \text{Out}(\mathcal{R})$. Since $\mathcal{U} = \mathcal{R}_{\Gamma}$ we must have $\tilde{\Gamma} \leq [\mathcal{U}]$ and therefore $\tilde{\Gamma} \leq N_{\mathcal{U}}(\mathcal{R})$ which implies $\Gamma \leq N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}]$.

On the other hand, we clearly have $\mathcal{R}_{N_{\mathcal{U}}(\mathcal{R})} \leq \mathcal{U} = \mathcal{R}_{\Gamma}$. So Lemma 3.2 implies $N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}] \leq \Gamma$. Theorem 3.3 now implies $N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}] = \Gamma \cong \mathcal{U}/\mathcal{R}$. □

4 Generalized Bernoulli shifts and cocycle superrigidity

Let G be a countable group, I a countable set on which G acts and (X_0, μ_0) a standard probability space. We let X_0^I be the set of all functions $x : I \rightarrow X_0$ and μ_0^I the product measure on X_0^I . Then $G \curvearrowright X_0^I$ by $(gx)(i) = x(g^{-1}i)$. This action preserves the measure μ_0^I . The action $G \curvearrowright (X_0, \mu_0)^I$ is a **generalized Bernoulli shift**.

Our interest in these actions stems from Popa's Cocycle Super-rigidity Theorem. To explain, let $G \curvearrowright (X, \mu)$ be a probability-measure-preserving action. A **cocycle** into a countable group H is a Borel map $c : G \times X \rightarrow H$ such that

$$c(g_1 g_2, x) = c(g_1, g_2 x) c(g_2, x).$$

Alternatively, if $G \curvearrowright (X, \mu)$ is essentially free then we can identify $G \times X$ with the orbit-equivalence relation, denoted \mathcal{R} , via $(g, x) \mapsto (gx, x)$. In this way, we can think of the cocycle as map from \mathcal{R} to H . We say the action is **cocycle superrigid** if for every such cocycle there is a homomorphism $\rho : G \rightarrow H$ and a Borel map $\phi : X \rightarrow H$ such that

$$c(g, x) = \phi(gx)^{-1} \rho(g) \phi(x).$$

The next result is a special case of celebrated theorem due to S. Popa [Po07] (see also [PV08, Theorem 3.2 and Proposition 2.3]).

Theorem 4.1. *Suppose every orbit of $G \curvearrowright I$ is infinite. If G has property (T) then the generalized Bernoulli shift $G \curvearrowright (X_0, \mu_0)^I$ is cocycle superrigid.*

5 A simple equivalence relation

Theorem 5.1. *Suppose $G \curvearrowright (X, \mu)$ is an essentially free measure-preserving ergodic action of a countably infinite group G on a standard probability space (X, μ) . Let $\mathcal{R} = \{(x, gx) : x \in X, g \in G\}$ be the orbit equivalence relation. Suppose $G \curvearrowright (X, \mu)$ is cocycle superrigid. If G is simple then \mathcal{R} has no proper ergodic normal subequivalence relations. If G has no nontrivial finite quotients then \mathcal{R} has no proper ergodic normal finite-index subequivalence relations.*

Proof. Let \mathcal{R} be the orbit-equivalence relation of the action $G \curvearrowright (X, \mu)$. Let $\mathcal{N} \triangleleft \mathcal{R}$ be an ergodic normal subequivalence relation and $c : \mathcal{R} \rightarrow \mathcal{R}/\mathcal{N}$ the canonical cocycle. Since the action is cocycle superrigid, there exists a Borel map $\phi : X \rightarrow H := \mathcal{R}/\mathcal{N}$ and a homomorphism $\rho : G \rightarrow H$ such that

$$c(gx, x) = \phi(gx)^{-1} \rho(g) \phi(x).$$

Claim 1. If ρ is trivial then $\mathcal{N} = \mathcal{R}$.

Proof of Claim 1. For every $g \in \mathcal{R}/\mathcal{N}$, $\phi^{-1}(g) \subset X$ is \mathcal{N} -invariant. Because \mathcal{N} is ergodic, this implies ϕ is essentially constant which implies $\mathcal{N} = \mathcal{R}$. \square

Claim 2. ρ is non-injective.

Proof of Claim 2. To obtain a contradiction, suppose ρ is injective. We claim that \mathcal{N} is finite. To see this, let

$$X_g = \{x \in X : (gx, x) \in \mathcal{N}\} = \{x \in X : \phi(gx)\phi(x)^{-1} = \rho(g)\}.$$

Also, for $g \in G, h \in \mathcal{R}/\mathcal{N}$ let

$$X_{g,h} = \{x \in X_g : \phi(x) = h\} = \{x \in X : \phi(x) = h, \phi(gx) = \rho(g)h\}.$$

Because ρ is injective, for any fixed h , the sets $\{gX_{g,h} : g \in G\}$ are pairwise disjoint. Therefore $\sum_{g \in G} \mu(X_{g,h}) \leq 1$. By the Borel-Cantelli Lemma, almost every x is contained in at most finitely many of the $X_{g,h}$'s (for fixed h). However for each $x \in X$ there is at exactly one h such that x is contained in some $X_{g,h}$. So, in fact, x is contained in at most finitely many $X_{g,h}$'s (allowing g and h to vary). Since $X_g = \cup_h X_{g,h}$ this implies that a.e. x is contained in at most finitely many X_g 's which implies that for a.e. x , the \mathcal{N} -equivalence class $[x]_{\mathcal{N}}$ is finite.

Because G is infinite and $G \curvearrowright (X, \mu)$ is essentially free and ergodic, μ is nonatomic. Because \mathcal{N} is finite and μ is nonatomic, \mathcal{N} is not ergodic. This contradiction proves that ρ is non-injective. \square

If G is simple, either ρ is trivial or injective and so the claims above finish the proof. If G does not have any nontrivial finite quotients and \mathcal{R}/\mathcal{N} is finite then $\rho : G \rightarrow \mathcal{R}/\mathcal{N}$ must be trivial. So Claim 1 finishes the proof. \square

Definition 2 (Compressions). Let $\mathcal{R} \subset X \times X$ be a probability-measure-preserving Borel equivalence relation on a probability space (X, μ) . If $Y \subset X$ has positive measure then we let $\mathcal{R}_Y := \mathcal{R} \cap (Y \times Y)$ denote the **compression of \mathcal{R} by Y** . It is an equivalence relation on Y .

Lemma 5.2. *Let $Y \subset X$ be a Borel set with positive measure. Let $\mathcal{S} \leq \mathcal{R}_Y$ be a subequivalence relation. Then there exists a subequivalence relation $\mathcal{T} \leq \mathcal{R}$ such that $\mathcal{T}_Y = \mathcal{S}$. Moreover, if \mathcal{S} is ergodic then \mathcal{T} is ergodic and if \mathcal{S} is normal in \mathcal{R}_Y then \mathcal{T} is normal in \mathcal{R} .*

Remark 2. \mathcal{T} is not unique. Moreover, even if \mathcal{S} is ergodic and normal there may exist subequivalence relations \mathcal{T}' such that $\mathcal{T}'_Y = \mathcal{S}$ but \mathcal{T}'_Y is neither ergodic nor normal.

Proof. Because \mathcal{R} is ergodic there exists a Borel map $\phi : X \rightarrow Y$ with graph contained in \mathcal{R} such that $\phi(y) = y$ for every $y \in Y$. Define the subequivalence relation \mathcal{T} by $x\mathcal{T}y$ iff $\phi(x)\mathcal{S}\phi(y)$. In other words, if $\Phi : \mathcal{R} \rightarrow \mathcal{R}_Y$ is the map $\Phi(x, y) = (\phi(x), \phi(y))$ then $\mathcal{T} = \Phi^{-1}(\mathcal{S})$. This implies that \mathcal{T} is Borel. It is easy to check that \mathcal{T} is a subequivalence relation and $\mathcal{T}_Y = \mathcal{S}$.

Suppose that \mathcal{S} is ergodic. Let $A \subset X$ be an \mathcal{T} -saturated set of positive measure. Observe that $A = \phi^{-1}(\phi(A))$ by definition of \mathcal{T} . Also $\phi(A)$ is \mathcal{S} -saturated. Therefore $\phi(A) = Y$ since \mathcal{S} is ergodic. So $A = \phi^{-1}(\phi(A)) = X$. Because A is arbitrary, \mathcal{T} is ergodic.

Suppose that \mathcal{S} is normal. Then there exists a groupoid morphism $c : \mathcal{R}_Y \rightarrow \mathcal{G}$ such that $\mathcal{S} = \text{Ker}(c)$. Define $c' : \mathcal{R} \rightarrow \mathcal{G}$ by $c'(x, y) = c(\phi(x), \phi(y))$. Observe that

$$c'(x, y)c'(y, z) = c(\phi(x), \phi(y))c(\phi(y), \phi(z)) = c(\phi(x), \phi(z)) = c'(x, z).$$

So c' is a cocycle. If $(x, y) \in \text{Ker}(c')$ then $c(\phi(x), \phi(y)) \in \mathcal{G}^0$ which implies $(\phi(x), \phi(y)) \in \text{Ker}(c) = \mathcal{S}$ which implies $(x, y) \in \mathcal{T}$. So $\text{Ker}(c') \subset \mathcal{T}$. On the other hand, if $(x, y) \in \mathcal{T}$ then $(\phi(x), \phi(y)) \in \mathcal{S} = \text{Ker}(c)$ which implies $(x, y) \in \text{Ker}(c')$. So $\mathcal{T} = \text{Ker}(c')$ is normal by Theorem 2.1. □

Proposition 5.3. *Let \mathcal{R} be an ergodic probability-measure-preserving equivalence relation with a finite-index ergodic subequivalence relation $\mathcal{S} \leq \mathcal{R}$. Then \mathcal{R} has a finite-index ergodic normal subequivalence relation \mathcal{N} with $\mathcal{N} \leq \mathcal{S}$.*

Proof. Let $n = [\mathcal{R} : \mathcal{S}]$ denote the index of \mathcal{S} in \mathcal{R} . Let $\phi : \mathcal{R} \rightarrow \{1, \dots, n\}$ be any Borel function satisfying

- for a.e. $x \in X$, $\phi(x, x) = 1$
- for a.e. $(x, y), (x, z) \in \mathcal{R}$ with $(y, z) \in \mathcal{S}$, $\phi(x, y) = \phi(x, z)$
- for a.e. $x \in X$, the map $y \mapsto \phi(x, y)$ surjects onto $\{1, \dots, n\}$. So this map is a bijection from the set of \mathcal{S} -classes in $[x]_{\mathcal{R}}$ to $\{1, \dots, n\}$.

Define a cocycle $\alpha : \mathcal{R} \rightarrow \text{Sym}(n)$ (the symmetric group of $\{1, \dots, n\}$) by

$$\alpha(x, y)(k) = \phi(x, z)$$

where $z \in [x]_{\mathcal{R}}$ is any element satisfying $\phi(y, z) = k$. Let \mathcal{K} be the kernel of this cocycle. This is a finite-index normal subequivalence relation and $\mathcal{K} \leq \mathcal{S}$ but \mathcal{K} might not be ergodic. However, it can have at most finitely many ergodic components (this is true for any finite-index subequivalence relation). Let $Y \subset X$ be an ergodic component of \mathcal{K} . So \mathcal{K}_Y is an ergodic finite-index normal subequivalence relation of \mathcal{R}_Y . By Lemma 5.2 there exists an ergodic normal finite index subequivalence relation $\mathcal{N} \leq \mathcal{R}$ such that $\mathcal{N}_Y = \mathcal{K}_Y$. Since \mathcal{N} and \mathcal{S} are ergodic and $\mathcal{N}_Y \leq \mathcal{S}_Y$ we must have that $\mathcal{N} \leq \mathcal{S}$. □

Proof of Theorem 1.1. Let G be a simple property (T) group. Quoting from [Th10]: there are two sources of simple groups with Kazhdan's property (T). Such groups appear for example as lattices in certain Kac-Moody groups, see [CR06]. Much earlier, it was also shown by Gromov ([Gr87]) that every hyperbolic group surjects onto a Tarski monster, i.e. every proper subgroup of this quotient is finite cyclic; in particular: this quotient group is simple and is a Kazhdan group if the hyperbolic group was a Kazhdan group.

Let (X_0, μ_0) be a nontrivial Borel probability space and $G \curvearrowright (X, \mu) := (X_0, \mu_0)^G$ the Bernoulli shift action. By Popa's Cocycle Superrigidity Theorem 4.1, $G \curvearrowright (X, \mu)$ is cocycle superrigid. So Theorem 5.1 implies \mathcal{R} has no ergodic proper normal subequivalence relations. Proposition 5.3 implies \mathcal{R} has no ergodic proper finite-index subequivalence relations □

6 Treeable equivalence relations

Definition 3. A **graphing** of an equivalence relation $\mathcal{R} \subset X \times X$ is a Borel subset $G \subset X \times X$ such that \mathcal{R} is the smallest equivalence relation containing G and G is symmetric: $(x, y) \in G \Rightarrow (y, x) \in G$. The **local graph** of G at x is denoted by G_x . It has vertex set $[x]_{\mathcal{R}}$ and edges $\{y, z\}$ where $y, z \in [x]_{\mathcal{R}}$ and $(y, z) \in G$. So G is a graphing if and only if it is symmetric and all local graphs are connected. A graphing is a **treeing** if all of its local graphs are trees.

Definition 4. Let \mathcal{R} be an ergodic treeable equivalence relation. A subequivalence relation $\mathcal{S} \leq \mathcal{R}$ is **primitive** if there exist treeings $G_{\mathcal{S}}, G_{\mathcal{R}}$ of \mathcal{S} and \mathcal{R} such that $G_{\mathcal{S}} \subset G_{\mathcal{R}}$. This means the same as free factor as used in [Ga00, Ga05].

Example 1. If $F = \langle S \rangle$ is a free group with free generating set $S \subset F$ and $F \curvearrowright (X, \mu)$ is an essentially free action then $G_F = \{(x, sx), (sx, x) : x \in X, s \in S\}$ is a treeing of the orbit-equivalence relation \mathcal{R} . Moreover if $g \in S$ and \mathcal{S} is the orbit-equivalence relation generated by $\{g^n\}_{n \in \mathbb{Z}}$ then \mathcal{S} is primitive in \mathcal{R} since $G_{\mathcal{S}} = \{(x, gx), (gx, x) : x \in X\}$ is a treeing of \mathcal{S} and $G_{\mathcal{S}} \subset G_F$. More generally, if g is primitive in F (this means that it is contained in some free generating set of F) and \mathcal{S} is the orbit-equivalence relation of $\{g^n\}_{n \in \mathbb{Z}}$, then \mathcal{S} is primitive in \mathcal{R} .

Before proving Theorem 1.2 we need a lemma.

Lemma 6.1. *Suppose Γ is a countable group and $c : \mathcal{R} \rightarrow \Gamma$ is a cocycle such that $\text{Ker}(c) \leq \mathcal{R}$ is ergodic. Let*

$$\Lambda = \{g \in \Gamma : \mu_L(\{(x, y) \in \mathcal{R} : c(x, y) = g\}) > 0\}.$$

Then Λ is a subgroup of Γ and $\mathcal{R}/\text{Ker}(c)$ is isomorphic to Λ .

Proof. For $x \in X$, let $\Gamma_x = \{c(x, y) : y \in [x]_{\mathcal{R}}\}$. If $(x, z) \in \text{Ker}(c)$ then $c(x, y) = c(z, y)$. So $\Gamma_x = \Gamma_z$. Since $\text{Ker}(c)$ is ergodic, this implies the existence of a subset $\Gamma' \subset \Gamma$ such that $\Gamma' = \Gamma_x$ for a.e. x . Observe that since $c(x, y) \in \Gamma_x$, $c(y, x) = c(x, y)^{-1} \in \Gamma_y$. Thus Γ' is invariant under inverse. Also if $c(x, y) \in \Gamma_x$ and $c(y, z) \in \Gamma_y$ then $c(x, z) = c(x, y)c(y, z) \in \Gamma_x$. So Γ' is a subgroup. By ergodicity again, $\Lambda = \Gamma'$. So without loss of generality, we may assume $\Lambda = \Gamma$.

It follows from [FSZ89, Theorem 2.2] that there is a homomorphism $\theta' : \mathcal{R}/\text{Ker}(c) \rightarrow \Gamma$ such that if $\theta : \mathcal{R} \rightarrow \mathcal{R}/\text{Ker}(c)$ is the canonical morphism then

$$\theta'\theta = c.$$

Since c and θ have the same kernel, θ' must be injective. Since $\Lambda = \Gamma$, it is also surjective and so $\mathcal{R}/\text{Ker}(c)$ is isomorphic to Λ . □

Theorem 6.2. *Suppose \mathcal{R} is a treeable ergodic probability-measure-preserving equivalence relation on (X, μ) of cost > 1 and there exists a subequivalence relation $\mathcal{S} \leq \mathcal{R}$ that is primitive, ergodic and proper. Then \mathcal{R} surjects onto every countable group.*

Proof. Because $\mathcal{S} \leq \mathcal{R}$ is primitive, there exist treeings $G_{\mathcal{S}} \subset G_{\mathcal{R}}$ of \mathcal{S} and \mathcal{R} . Because \mathcal{S} is proper, $\mu_L(\mathcal{R} \setminus \mathcal{S}) > 0$ and therefore $\mu_L(G_{\mathcal{R}} \setminus G_{\mathcal{S}}) > 0$. Let $c : G_{\mathcal{R}} \setminus G_{\mathcal{S}} \rightarrow \mathbb{F}_{\infty}$ be any measurably surjective map such that $c(x, y) = c(y, x)^{-1}$ wherever this is defined. Here \mathbb{F}_{∞} denotes the free group of countable rank. We extend c to $G_{\mathcal{S}}$ by $c(x, y) = e$ for any $(x, y) \in \mathcal{S}$. Now c is defined on all of $G_{\mathcal{R}}$. Because $G_{\mathcal{R}}$ is a treeing there is a unique extension of c to a cocycle $c : \mathcal{R} \rightarrow \mathbb{F}_{\infty}$.

By definition $\text{Ker}(c)$ contains \mathcal{S} . Because \mathcal{S} is ergodic, this implies $\text{Ker}(c)$ is ergodic. Lemma 6.1 now implies $\mathcal{R}/\text{Ker}(c) \cong \mathbb{F}_{\infty}$.

Now let Λ be an arbitrary countable group and $\phi : \mathbb{F}_{\infty} \rightarrow \Lambda$ a surjective homomorphism. Let $c' : \mathcal{R} \rightarrow \Lambda$ be the cocycle $c'(x, y) = \phi(c(x, y))$. Since $\text{Ker}(c)$ is ergodic, $\text{Ker}(c')$ is ergodic. So Lemma 6.1 implies $\mathcal{R}/\text{Ker}(c') \cong \Lambda$. □

Example 2. If \mathcal{R} is the orbit-equivalence relation of a Bernoulli shift action of \mathbb{F}_n ($n \geq 2$) then every generator of \mathbb{F}_n acts ergodically. Therefore, \mathcal{R} satisfies the hypotheses of Theorem 6.2.

Remark 3. Clinton Conley, Damien Gaboriau, Andrew Marks and Robin Tucker-Drob have proven that any treeable *strongly ergodic* pmp equivalence relation satisfies the hypothesis of Theorem 6.2. This work has not yet been published.

Conjecture 1. *Let \mathcal{R} be an ergodic treeable equivalence relation of cost > 1 . Then there exists an ergodic element $f \in [\mathcal{R}]$ such that the subequivalence relation generated by f is primitive in \mathcal{R} .*

It follows from Theorem 6.2 that if the above conjecture is true then every treeable ergodic probability-measure-preserving equivalence relation surjects every countable group.

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